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The Asymptotic Theory of Global Solutions for Semilinear Wave Equations in Three Space Dimensions

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Abstract—This paper deals with the asymptotic theory of initial value problems for semilinear wave equations in three space dimensions. The well posedness and validity of formal approximations about time $T = \infty$ are discussed in the classical sense of C^2. The results describe the validity of formal global solutions. Using a time-scale perturbation method, an application of the asymptotic theory is given to analyze a special wave equation in three space dimensions.

Keywords—Semilinear wave equation, Asymptotes of global solutions, Application.

1. INTRODUCTION

In this paper, an asymptotic theory of global solutions is established for the following initial value problem consisting of a semilinear perturbed wave equation and two initial conditions

$$u_{tt} - \Delta u = \varepsilon f(u, \varepsilon), \qquad t > 0, \quad x \in R^3, \tag{1}$$

$$u(0, x, \varepsilon) = u_0(x, \varepsilon), \qquad x \in R^3, \tag{2}$$

$$u_t(0, x, \varepsilon) = u_1(x, \varepsilon), \qquad x \in R^3, \tag{3}$$

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where u is a real-valued unknown function, $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$, ε is a parameter with $0 < |\varepsilon| \le \varepsilon_0 \ll 1$, $f(u,\varepsilon), u_0(x,\varepsilon)$, and $u_1(x,\varepsilon)$ satisfy some assumptions to be given in Section 2. The asymptotic theory of global solutions implies the well posedness (in the classical sense of C^2) of the initial value problem (1)–(3) and the asymptotic validity of formal global solutions.

In [1–4], the asymptotic theory for validation of the formal approximations of the solutions to the initial-boundary value problems associated with the second-order semilinear wave equation in one space dimension with the time order function $T = O(|\varepsilon|^{-1})$ has been presented. But for $x \in R^1$, there still exist some unsolved open problems, as given in [1–3], on the asymptotic theory of initial value problems associated with second-order nonlinear wave equations. The reason is that the study of asymptotic theory of initial value problems for partial differential equations is far more difficult than that of initial-boundary value problems. In [5], the asymptotic theory of solutions to the initial value problems associated with equation $u_{tt} - u_{xx} + p^2 u = \varepsilon f(t,x,u,\varepsilon)(-\infty < x < \infty,\ p^2 > 0)$ was developed on the long time scale of order $|\varepsilon|^{-1}$ in a suitable Sobolev space. For the nonlinear partial differential equations in high space dimensions, as stated in [2], only very little is known about the asymptotic theory. In [6], the asymptotic theory of initial value problems for second-order wave equations in the classical senses of C^2 is presented on the local time scale of order $|\varepsilon|^{-1}$ in three space dimensions. In this paper, the asymptotic theory and its application to the validation of the formal global approximation to the solution of the problem governed by the second-order semilinear wave equation (1) and conditions (2) and (3) are established in the classical sense of C^2 on the time $T_\Delta t = \infty$. This result describes the asymptotic behavior of the global solutions of the problem defined by (1)–(3).

The rest of the paper is organized as follows. In Section 2, the well posedness of the initial value problems (1)–(3) for global solutions is presented in the classical sense of C^2. The asymptotic validation of formal global approximations is established in Section 3. As an application of the asymptotic theory, a special perturbed wave equation in three space dimensions is analyzed in Section 4 by using a time-scale perturbation method, which indicates an asymptotic form for the solution that decays like $(1+t+|x|)^{-k}$ $(0<k<1)$.

2. THE WELL POSEDNESS

In order to prove the existence and uniqueness of the solution, in the classical sense of C^2, for problem (1)–(3), we developed, based on the work presented in [7], the following equivalent integral equation for problem (1)–(3)

$$\begin{aligned} u(t,x,\varepsilon) &= \left\{ \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\xi|=1} u_0(x+t\xi,\varepsilon)\,d\sigma_\xi\right] + \frac{t}{4\pi}\int_{|\xi|=1} u_1(x+t\xi,\varepsilon)\,d\sigma_\xi \right\} \\ &\quad + \varepsilon\int_0^t \frac{t-\tau}{4\pi}\int_{|\xi|=1} f(u(\tau, x+(t-\tau)\xi,\varepsilon),\varepsilon)\,d\sigma_\xi\,d\tau \\ &= u^0(t,x,\varepsilon) + v^0(t,x,\varepsilon), \end{aligned} \tag{4}$$

where ξ is a unit vector in R^3 and $d\sigma_\xi$ is an area element on a sphere of radius 1.

Suppose that the nonlinear term f and the initial values u_0 and u_1 satisfy the following assumptions.

(i) $f(u,\varepsilon) \in C^2$ with respect to $u, f(0,\varepsilon) = f_u(0,\varepsilon) = f_{uu}(0,\varepsilon) = 0$.

(ii) If $|u(t,x,\varepsilon)| < M, |v(t,x,\varepsilon)| < M$, there exist constants $p > 3$ and $A > 0$, such that

$$|f(u,\varepsilon)| \le A,$$
$$|f_{uu}(u,\varepsilon) - f_{uu}(v,\varepsilon)| \le A|\omega|^{p-3}|u-v|,$$

where $\omega = \max\{|u|,|v|\}$ and the constants M and A are independent of ε.

It should be addressed that the assumption of $f(u,\varepsilon)$ is essential for obtaining the estimates in Lemma 3.

(iii) $u_0(x,\varepsilon)$ and $u_1(x,\varepsilon)$ satisfy

$$\left|\partial_x^\alpha u_0(x,\varepsilon)\right|, \quad \left|\partial_x^\beta u_1(x,\varepsilon)\right| \le \frac{G}{(1+|x|)^{k+1}}, \qquad 0<k<1,$$

where the multi-integers α and β satisfy $|\alpha|\le 3$, $|\beta|\le 2$, G is independent of ε.

Let J be given by

$$J=\left\{(t,x)\mid t\ge 0,\ x\in R^3\right\}.$$

We define $C_R^2(J_k)$ be the space of all real-valued and twice continuously differentiable functions W on J_k with the norm $\|\cdot\|_{J_k}$ given by

$$\|W\|_{J_k}=\operatorname*{Sup}_{(t,x)\in J_k}\left[(1+t+|x|)^k\|W(t,x,\varepsilon)\|\right]<R \qquad (R>1), \tag{5}$$

where

$$\|W(t,x,\varepsilon)\|=\sum_{0\le j+i_1+i_2+i_3\le 2}\left|\frac{\partial^{j+i_1+i_2+i_3}W(t,x,\varepsilon)}{\partial t^j\partial x_1^{i_1}\partial x_2^{i_2}\partial x_3^{i_3}}\right|.$$

REMARK 1. For simplicity, throughout the paper, we denote by C any positive constants independent of ε and R, which may depend on k,p,M,A,G, and function f.

REMARK 2. By the definition of space $C_R^2(J_k)$, we know that $C_R^2(J_k)$ is a Banach space with the norm defined by (5). We shall use the fixed-point theorem to prove the existence and uniqueness of the solution to problem (1)–(3) in the space $C_R^2(J_k)$. Indeed, space $C_R^2(J_k)$ is a ball with radius R in real-valued and twice continuously differentiable functions.

LEMMA 1. *If* $0<k<1$, *then*

$$\frac{t}{4\pi}\int_{|\xi|=1}(1+|x+t\xi|)^{-k-1}\,d\sigma_\xi \le \frac{C}{(1+t+|x|)^k},$$

$$\frac{1}{4\pi}\int_{|\xi|=1}(1+|x+t\xi|)^{-k-1}\,d\sigma_\xi \le \frac{C}{(1+t+|x|)^k}.$$

The proof can be found in [8, pp. 1465–1466].

LEMMA 2. *Suppose that* $u_0(x,\varepsilon)$ *and* $u_1(x,\varepsilon)$ *satisfy (iii), then*

$$\left\|u^0(t,x,\varepsilon)\right\| \le \frac{C}{(1+t+|x|)^k}, \qquad 0<k<1.$$

The proof can be in [6, p. 323].

Let the operator $\wedge$ be defined as follows:

$$\begin{aligned}\wedge u &= \left\{\frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\xi|=1}u_0(x+t\xi,\varepsilon)\,d\sigma_\xi\right]+\frac{t}{4\pi}\int_{|\xi|=1}u_1(x+t\xi,\varepsilon)\,d\sigma_\xi\right\}\\ &\quad+\varepsilon\int_0^t\frac{t-\tau}{4\pi}\int_{|\xi|=1}f(u(\tau,x+(t-\tau)\xi,\varepsilon),\varepsilon)\,d\sigma_\xi\,d\tau\\ &= u^0(t,x,\varepsilon)+v^0(t,x,\varepsilon).\end{aligned}$$

LEMMA 3. *Suppose that f, u_0, and u_1 satisfy Assumptions (i)–(iii). Then, for any u, $v \in C_R^2(J_k)$, $p > 3$, and $k > 2/(p-1)$,*

(a)

$$\| \wedge u \| \le \frac{C}{(1+t+|x|)^k} + \frac{C|\varepsilon| R^{p-1} \|u\|_{J_k}}{(1+t+|x|)^k},$$

(b)

$$\| \wedge u - \wedge v \| \le \frac{C|\varepsilon| R^{p-1} \|u - v\|_{J_k}}{(1+t+|x|)^k}.$$

PROOF. Since

$$v^0(t,x,\varepsilon) = \varepsilon \int_0^t \frac{t-\tau}{4\pi} \int_{|\xi|=1} f(u(\tau, x+(t-\tau)\xi, \varepsilon), \varepsilon)\, d\sigma_\xi\, d\tau. \tag{6}$$

$$\begin{aligned} v_t^0(t,x,\varepsilon) = {} & \frac{\varepsilon}{4\pi} \int_0^t \int_{|\xi|=1} f(u(\tau, x+(t-\tau)\xi, \varepsilon), \varepsilon)\, d\sigma_\xi\, d\tau \\ & + \varepsilon \int_0^t \frac{t-\tau}{4\pi} \int_{|\xi|=1} f_u \cdot u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon) \cdot \xi\, d\sigma_\xi\, d\tau, \end{aligned} \tag{7}$$

where $v' = x + (t-\tau)\xi$,

$$\begin{aligned} v_{tt}^0(t,x,\varepsilon) = {} & \frac{\varepsilon}{4\pi} f(u(t,x,\varepsilon),\varepsilon) \int_{|\xi|=1} d\sigma_\xi \\ & + \frac{2\varepsilon}{4\pi} \int_0^t \int_{|\xi|=1} f_u \cdot u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon) \cdot \xi\, d\sigma_\xi\, d\tau \\ & + \varepsilon \int_0^t \frac{t-\tau}{4\pi} \int_{|\xi|=1} \{ f_{uu} \cdot u_{v'}^2(\tau, x+(t-\tau)\xi, \varepsilon)\xi^2 \\ & + f_u \cdot u_{v'v'}(\tau, x+(t-\tau)\xi, \varepsilon)\xi^2 \}\, d\sigma_\xi\, d\tau. \end{aligned} \tag{8}$$

It follows from Assumptions (i) and (ii) that

$$|f(u(t,x,\varepsilon))| \le C|u(t,x)|^p \frac{(1+t+|x|)^{kp}}{(1+t+|x|)^{kp}} \le \frac{CR^{p-1}\|u\|_{J_k}}{(1+t+|x|)^{kp}}, \tag{9}$$

$$\begin{aligned} |f_u \cdot u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon)| & \le C|u(\tau, x+(t-\tau)\xi, \varepsilon)|^{p-1} |u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon)| \\ & = \frac{C\left[(1+\tau+|x+(t-\tau)\xi|)^k |u(\tau, x+(t-\tau)\xi)|\right]^{p-1}}{(1+\tau+|x+(t-\tau)\xi|)^{kp-k}} \\ & \quad \times \frac{\left[(1+\tau+|x+(t-\tau)\xi|)^k |u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon)|\right]}{(1+\tau+|x+(t-\tau)\xi|)^k} \\ & \le \frac{C\|u\|_{J_k}^{p-1} \|u\|_{J_k}}{(1+\tau+|x+(t-\tau)\xi|)^{kp}} \\ & \le \frac{CR^{p-1}\|u\|_{J_k}}{(1+\tau+|x+(t-\tau)\xi|)^{kp}}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \left| f_{uu} \cdot u_{v'}^2(\tau, x+(t-\tau)\xi, \varepsilon) \right| & \le C|u(\tau, x+(t-\tau)\xi)|^{p-2} |u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon)|^2 \\ & = \frac{C\left[(1+\tau+|x+(t-\tau)\xi|)^k |u(\tau, x+(t-\tau)\xi, \varepsilon)|\right]^{p-2}}{(1+\tau+|x+(t-\tau)\xi|)^{kp-2k}} \\ & \quad \times \frac{\left[(1+\tau+|x+(t-\tau)\xi|)^k |u_{v'}(\tau, x+(t-\tau)\xi, \varepsilon)|\right]^2}{(1+\tau+|x+(t-\tau)\xi|)^{2k}} \\ & \le \frac{C\|u\|_{J_k}^{p-2} \|u\|_{J_k}^2}{(1+\tau+|x+(t-\tau)\xi|)^{kp}} \\ & \le \frac{CR^{p-1}\|u\|_{J_k}}{(1+\tau+|x+(t-\tau)\xi|)^{kp}}. \end{aligned} \tag{11}$$

By the same estimate as that of (10) or (11), we have

$$|f_u u_{v'v'}(\tau, x+(t-\tau)\xi, \varepsilon)| \le \frac{CR^{p-1}\|u\|_{J_k}}{(1+\tau+|x+(t-\tau)\xi|)^{kp}}. \tag{12}$$

It follows from inequalities (6)–(12) that

$$\begin{aligned}\sum_{i=0}^{2}\left|\frac{\partial^i v^0(t,x,\varepsilon)}{\partial t^i}\right| &\le \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^{kp}}\\ &\quad + C|\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t (t-\tau)\int_{|\xi|=1}\frac{d\sigma_\xi\, d\tau}{(1+\tau+|x+(t-\tau)\xi|)^{kp}}\\ &\quad + C|\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t \int_{|\xi|=1}\frac{d\sigma_\xi\, d\tau}{(1+\tau+|x+(t-\tau)\xi|)^{kp}}\\ &= I_1+I_2+I_3.\end{aligned}$$

By Lemma 1, we get

$$\begin{aligned} I_2 &\le C|\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t \frac{t-\tau}{(1+\tau)^{kp-(k+1)}}\int_{|\xi|=1}\frac{d\sigma_\xi\, d\tau}{(1+\tau+|x+(t-\tau)\xi|)^{k+1}}\\ &= C\ |\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t \frac{1}{(1+\tau)^{kp-1}}\cdot\frac{t-\tau}{(1+\tau)}\int_{|\xi|=1}\frac{d\sigma_\xi\, d\tau}{(1+|x/(1+\tau)+(t-\tau)/(1+\tau)\xi|)^{k+1}}\\ &\le C|\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t \frac{1}{(1+\tau)^{kp-1}}\cdot\frac{d\tau}{(1+(t-\tau)/(1+\tau)+|x|/(1+\tau))^{k}}\\ &= C|\varepsilon|R^{p-1}\|u\|_{J_k}\int_0^t \frac{d\tau}{(1+\tau)^{kp-(k+1)}}\cdot\frac{1}{(1+\tau+|x|)^k}\\ &= \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k}\int_0^t\frac{d\tau}{(1+\tau)^{kp-(k+1)}}.\end{aligned}$$

Since $kp-(k+1)>1$ (namely, $k>2/(p-1)$), we conclude that

$$\int_0^\infty \frac{d\tau}{(1+\tau)^{kp-(k+1)}}$$

is convergent. Therefore, we have

$$I_2 \le \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k}, \qquad \text{if } k>\frac{2}{p-1}. \tag{13}$$

By the same estimate as that of I_2 and using Lemma 1, we have

$$I_3 \le \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+\tau+|x|)^k}, \qquad \text{if } k>\frac{2}{p-1}. \tag{14}$$

Since

$$I_1 = \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^{kp}} \le \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k}, \qquad \text{if } k>\frac{2}{p-1}. \tag{15}$$

It follows from (13)–(15) that

$$\sum_{i=0}^{2}\left|\frac{\partial v^0(t,x,\varepsilon)}{\partial t^j}\right| \le \frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k}, \qquad \text{if } k>\frac{2}{p-1}. \tag{16}$$

By the same estimate as that of (16), it follows that

$$\sum_{0\le j+i_1+i_2+i_3\le 2}\left|\frac{\partial^{j+i_1+i_2+i_3}v^0(t,x,\varepsilon)}{\partial t^j\partial x_1^{i_1}\partial x_2^{i_2}\partial x_3^{i_3}}\right|\le\frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k},\qquad \text{if } k>\frac{2}{p-1}. \tag{17}$$

From (17), we immediately know

$$\left\|v^0(t,x,\varepsilon)\right\|\le\frac{C|\varepsilon|R^{p-1}\|u\|_{J_k}}{(1+t+|x|)^k},\qquad \text{if } k>\frac{2}{p-1}. \tag{18}$$

It follows from Lemma 2 and (18) that Inequality (a) in Lemma 3 holds. Since $u,v\in C^2(J_k)$, we only need to substitute $u-v$ for u in the proof of inequality (18), and we can get

$$\|\wedge u-\wedge v\|\le\frac{C|\varepsilon|R^{p-1}\|u-v\|_{J_k}}{(1+t+|x|)^k},\qquad \text{if } k>\frac{2}{p-1}.$$

The proof of Lemma 3 is completed.

Now, we have the following well-posedness theorem.

THEOREM 1. *Suppose that the nonlinear term $f(u,\varepsilon)$, initial data $u_0(x,\varepsilon)$ and $u_1(x,\varepsilon)$ satisfy Assumptions (i)–(iii) with sufficiently small ε_0, such that $0<\varepsilon\le\varepsilon_0\le 1$, and $0<k<1$, then if $k>2/(p-1)(p>3)$, $t>0$, $x\in R^3$, there exists a unique global $C_R^2(J_k)$-solution to problem (1)–(3).*

PROOF. Let C be the constant as defined in Lemma 3, $u\in C_R^2(J_k)$, $R>2C$, and let ε_0 be sufficiently small such that $C|\varepsilon_0|R^{p-1}<1/2$, from Lemma 3, we can easily get for any $u,v\in C_R^2(J_k)$ that

$$\|\wedge u\|_{J_k}\le C+C|\varepsilon|R^{p-1}\|u\|_{J_k}<C+\frac{1}{2}R<\frac{1}{2}R+\frac{1}{2}R<R,\qquad \text{if } k>\frac{2}{p-1},$$
$$\|\wedge u-\wedge v\|_{J_k}\le C|\varepsilon|R^{p-1}\|u-v\|_{J_k}<\frac{1}{2}\|u-v\|_{J_k},\qquad \text{if } k>\frac{2}{p-1},$$

in which the operator $\wedge: C_R^2(J_k)\to C_R^2(J_k)$ is a contractive mapping. Therefore, there must exist a unique global solution in $C_R^2(J_k)$ to problem (1)–(3). The proof of Theorem 1 is completed.

3. ASYMPTOTIC VALIDATION OF GLOBAL SOLUTIONS

Since we know that the initial value problem (1)–(3) contains a small parameter ε, perturbation methods may be applied for the construction of approximations to the solutions. In many perturbation methods for nonlinear problems, a function is constructed in such a way that it satisfies the differential equation and initial conditions up to some order of ε (where the parameter ε is small). Such a function is usually called a formal approximation. In order to prove that the formal approximation is an asymptotic approximation (as $\varepsilon\to 0$), we have to carry out an additional analysis in the space $C_R^2(J_k)$.

Suppose that on $J\times[-\varepsilon_0,\varepsilon_0]$, the function $v(t,x,\varepsilon)$ satisfies

$$v_{tt}-\Delta v=\varepsilon f(v,\varepsilon)+|\varepsilon|^m c_1(t,x,\varepsilon),\qquad m>1, \tag{19}$$
$$v(0,x,\varepsilon)=u_0(x,\varepsilon)+|\varepsilon|^{m-1}c_2(x,\varepsilon)=v_0(x,\varepsilon)0<|\varepsilon|\le\varepsilon_0\ll 1, \tag{20}$$
$$v_t(0,x,\varepsilon)=u_1(x,\varepsilon)+|\varepsilon|^{m-1}c_3(x,\varepsilon)=v_1(x,\varepsilon)0<|\varepsilon|\le\varepsilon_0\ll 1, \tag{21}$$

where $f(u,\varepsilon)$, $u_0(x,\varepsilon)$, and $u_1(x,\varepsilon)$ satisfy Assumptions (i)–(iii). Suppose that $c_1(t,x,\varepsilon)$, $c_2(x,\varepsilon)$, and $c_3(x,\varepsilon)$ satisfy the following conditions

$$c_1(t,x,\varepsilon)\in C^2(J_k),\qquad \|c_1(t,x,\varepsilon)\|\le\frac{C}{(1+t+|x|)^{kp}}, \tag{22}$$

$$|\partial_x^\alpha c_2(x,\varepsilon)|, |\partial_x^\beta c_3(x,\varepsilon)| \leq \frac{C}{(1+|x|)^{k+1}}, \qquad |\alpha| \leq 3, \quad |\beta| \leq 2, \quad 0 < k < 1. \tag{23}$$

It follows from Theorem 1 that the initial value problem governed by (19)–(21) has a unique global solution $v(t,x,\varepsilon) \in C_R^2(J_k)$. On the other hand, problems (19)–(21) can be transformed into the following equivalent equation

$$\begin{aligned} v(t,x,\varepsilon) = {} & \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\xi|=1} v_0(x+t\xi,\varepsilon)\,d\sigma_\xi\right] + \frac{t}{4\pi}\int_{|\xi|=1} v_1(x+t\xi,\varepsilon)\,d\sigma_\xi \\ & + \varepsilon \int_0^t \frac{t-\tau}{4\pi}\int_{|\xi|=1} [f(v(\tau, x+(t-\tau)\xi,\varepsilon),\varepsilon) + |\varepsilon|^m c_1(\tau, x+(t-\tau)\xi,\varepsilon)]\,d\sigma_\xi\,d\tau. \end{aligned}$$

If $u \in C_R^2(J_k)$ is the solution of problem (1)–(3), then

$$\begin{aligned} v(t,x,\varepsilon) - u(t,x,\varepsilon) = {} & \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\xi|=1} |\varepsilon|^{m-1} c_2(x+t\xi,\varepsilon)\,d\sigma_\xi\right] \\ & + \frac{t}{4\pi}\int_{|\xi|=1} |\varepsilon|^{m-1} c_3(x+t\xi,\varepsilon)\,d\sigma_\xi \\ & + \varepsilon\int_0^t \frac{t-\tau}{4\pi}\int_{|\xi|=1} \{[f(v(\tau,x+(t-\tau)\xi,\varepsilon),\varepsilon) \\ & - f(u(\tau,x+(t-\tau)\xi,\varepsilon),\varepsilon)] + |\varepsilon|^m c_1(\tau,x+(t-\tau)\xi,\varepsilon)\}\,d\sigma_\xi\,d\tau. \end{aligned} \tag{24}$$

Noting assumptions (22), (23), and Lemma 1, and by the same proof as that of Lemma 3, we have

$$\|v(t,x,\epsilon) - u(t,x,\epsilon)\| \leq \frac{C|\varepsilon|R^{p-1}\|u-v\|_{J_k} + C|\varepsilon|^{m-1}}{(1+t+|x|)^k}, \quad \text{if } k > \frac{2}{p-1}.$$

Choosing ε sufficiently small such that $C|\varepsilon|R^{p-1} < 1/2$, we have

$$\|v(t,x,\varepsilon) - u(t,x,\varepsilon)\|_{J_k} = O\left(|\varepsilon|^{m-1}\right).$$

Now, we can have the following asymptotic theorem of the global solution for problems (1)–(3).

THEOREM 2. *Suppose that $v(t,x,\varepsilon)$ is the solution of problems (19)–(21), and that the nonlinear term f and the initial data u_0 and u_1 satisfy Assumptions (i)–(iii). Let $c_1(t,x,\varepsilon)$, $c_2(x,\varepsilon)$, and $c_3(x,\varepsilon)$ satisfy (22) and (23). Then, for $m > 1$, the formal approximation $v(t,x,\varepsilon)$ is an asymptotic approximation (as $\varepsilon \to 0$) of the global solution $u(t,x,\varepsilon)$ of the problem defined by (1)–(3). Furthermore,*

$$\|u-v\|_{J_k} = O\left(|\varepsilon|^{m-1}\right), \qquad \text{for any } t \geq 0,\ x \in R^3, \quad \text{if } \frac{2}{p-1} < k < 1 \quad (p > 3).$$

4. EXAMPLE

In this section, an asymptotic approximation of the solution to the initial value problems (1)–(3) with $f(t,u,\varepsilon) \equiv u^4$ will be analyzed by using a time-scale perturbation method. The following initial value problem will be considered.

$$u_{tt} - \Delta u = \varepsilon u^4, \qquad t \geq 0, \quad x \in R^3, \quad 0 < |\varepsilon| \leq \varepsilon_0 \ll 1, \tag{25}$$

$$u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \qquad x \in R^3, \tag{26}$$

where

$$|\partial_x^\alpha \varphi(x)|, |\partial_x^\beta \psi(x)| \leq \frac{1}{(1+|x|)^{1+k}}, \qquad 0 < k < 1, \quad |\alpha| \leq 3, \quad |\beta| \leq 2.$$

Since the formal approximation will be constructed in the form of an infinite series, we require that the initial data $\varphi(x)$ and $\psi(x)$ are sufficiently smooth to get a convergent series representation for which summation and differentiation may be interchanged and all assumptions in Theorem 1 and Theorem 2 are satisfied in order to analyze the globally asymptotic approximation of problem (25) and (26). In the sense of Theorem 2, function $\bar{u}(t,x)$ will be constructed in such a way that it satisfies (25) and (26) up to order ε^2. From Theorem 2, it follows that $\|u(t,x)-\bar{u}(t,x)\|_{J_k} = O(|\varepsilon|)$ as $\varepsilon \to 0$ for $(t,x) \in J$. To construct $\bar{u}$, a perturbation method will be used. Assuming that $u(t,x,\varepsilon)$ has the following expansion

$$u(t,x,\varepsilon) = g_0(t,x) + \varepsilon g_1(t,x) + \varepsilon^2 g_2(t,x) + \cdots, \tag{27}$$

we will show that $u(t,x,\varepsilon)$ decays like $(1+t+|x|)^{-k}$ as $t \to \infty$, or $|x| \to \infty (\varepsilon \to 0)$. By substituting expansion (27) into (25), (26) and equating the coefficients of like power in ε, it follows from the powers 0 and 1 of ε, respectively, that g_0 should satisfy

$$g_{0_{tt}} - \Delta g_0 = 0, \tag{28}$$

$$g_0(x,0) = \varphi(x), \tag{29}$$

$$g_{0_t}(x,0) = \psi(x), \tag{30}$$

and that g_1 should satisfy

$$g_{1_{tt}} - \Delta g_1 = g_0^4(t,x), \tag{31}$$

$$g_1(x,0) = 0, \tag{32}$$

$$g_{1_t}(x,0) = 0. \tag{33}$$

Since the function $g_0(t,x)$ and $g_1(t,x)$ can be easily determined by (4), let $\bar{u}(t,x,\varepsilon) = g_0(t,x) + \varepsilon g_1(t,x)$, it follows

$$\begin{aligned} \bar{u}_{tt} - \Delta\bar{u} - \varepsilon\bar{u}^4 &= \left(g_{0_{tt}} - \Delta g_0\right) + \varepsilon\left(g_{1_{tt}} - \Delta g_1 - g_0^4\right) \\ &\quad + \varepsilon^2\left(4g_0^3 g_1 + 6\varepsilon g_0^2 g_1^2 + \cdots + \varepsilon^3 g_1^4\right) \\ &= \varepsilon^2 c_1^{\cdot}(t,x,\varepsilon), \end{aligned}$$

here $c_1^{\cdot}(t,x,\varepsilon)$ satisfies the assumption of Theorem 2. It follows from Theorem 2, that $\bar{u}(t,x)$ is an order ε asymptotic approximation (as $\varepsilon \to 0$) of the solution $u(t,x)$ of the initial value problem defined by (25) and (26) for $(t,x) \in J$, that is

$$\|u(t,x) - \bar{u}(t,x)\|_{J_k} = O\left(|\varepsilon|\right), \qquad \text{for } t > 0,\ x \in R^3. \tag{34}$$

Using (34), the following estimate can be obtained

$$\|u - g_0\|_{J_k} = \|u - \bar{u} + \bar{u} - g_0\|_{J_k} \le \|u - \bar{u}\|_{J_k} + \|\varepsilon g_1\|_{J_k} = O\left(|\varepsilon|\right), \qquad \text{on } J_k.$$

Taking $2/3 < k < 1$, which satisfies the assumption of Theorem 2, it follows from Theorem 2 that $\|u - \bar{u}\|_{J_k} = O(|\varepsilon|)$ and $\|u - g_0\|_{J_k} = O(|\varepsilon|)$ for any $(t,x) \in [0,\infty] \times R^3$. Moreover, we have

$$\|u\|_{J_k} \le \|g_0\|_{J_k} + O\left(|\varepsilon|\right) < C, \tag{35}$$

and then,

$$\|u\| < \frac{C}{(1+t+|x|)^k}. \tag{36}$$

Inequality (36) means u decays like $(1+t+|x|)^{-k}$ as $t \to \infty$ or $x \to \infty (\varepsilon \to 0)$.

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